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TAYLOR'S THEOREM AND ITS LIMIT.

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SUPPOSE φx and $\varphi' x$ are finite and continuous functions for all values of x from $x = x'$ to $x = x'+h$.

Then $$\frac{\varphi(x'+h)-\varphi x'}{h} = \varphi'(x'+\psi h),$$
where ψ is a positive proper fraction.

Let A and B be the algebraically greatest and least values of $\varphi' x$ between the values x' and $x'+h$ of x. Then A, B, $\varphi' x$ and $\frac{\varphi(x'+h)-\varphi x'}{h}$ are all positive, or all negative accordingly as φx is an increasing or a decreasing function.

Put $$y = Ax - \varphi x \quad (1), \qquad z = \varphi x - Bx. \quad (2)$$

Then $\frac{dy}{dx} = A - \varphi' x$, and $\frac{dz}{dx} = \varphi' x - B$, both of which are always positive, and consequently y and z are always increasing functions.

Let x take each of the two values x' and $x'+h$ in (1) and (2). Then,
$$y' = A(x'+h) - \varphi(x'+h),\quad z' = \varphi(x'+h) - B(x'+h),$$
$$y = Ax' - \varphi x', \text{ and } \quad z = \varphi x' - Bx'.$$ Therefore
$$\frac{y'-y}{h} = A - \frac{\varphi(x'+h)-\varphi x'}{h},\quad \frac{z'-z}{h} = \frac{\varphi(x'+h)-\varphi x'}{h} - B.$$
Because y and z are increasing functions, $y'-y$ and $z'-z$ are of the same sign as h, and therefore the members of the last two equations are always positive.

Consequently $[\varphi(x'+h)-\varphi x']\div h$ is always less than A and greater than B. But $\varphi' x$ in passing from A to B, passes through all intermediate values, and therefore through $[\varphi(x'+h)-\varphi x']\div h$; and in this value of $\varphi' x$, x has some value between x' and $x'+h$. Let $x'+\psi h$ represent this value of x, —ψ being a positive proper fraction. Then, $[\varphi(x'+h)-\varphi x']\div h = \varphi'(x'+\psi h)$; or we can write $[\varphi(x+h)-\varphi x]\div h = \varphi'(x+\psi h)$, which will be true for all values of x and h within the limits of finite and continuous values of φx and $\varphi' x$.

Suppose now that $\varphi x, \varphi' x, \varphi'' x \ldots \varphi^{2^n} x$ are finite and continuous functions, for all values of the variable from x to $x+h$; then by the foregoing principle we can have the following equations:

$[\varphi(x+h)-\varphi x]\div h = \varphi'(x+\psi_1 h)$, where ψ_1 is between 0 and 1. (*a*)
$[\varphi'(x+\psi_1 h)-\varphi' x]\div \psi_1 h = \varphi''(x+\psi_2 h)$, " ψ_2 " " 0 " ψ_1. (*b*)
$[\varphi''(x+\psi_2 h)-\varphi'' x]\div \psi_2 h = \varphi'''(x+\psi_3 h)$, " ψ_3 " " 0 " ψ_2. (*c*)

$[\varphi'''(x+\psi_3h)—\varphi'''x]\div\psi_3h=\varphi'''(x+\psi_4h)$, where ψ_4 is bet. 0 and ψ_3. (d)

. . . .

$\dfrac{\varphi^{n-1}(x+\psi_{n-1}h)-\varphi^{n-1}x}{\psi_{n-1}h}=\varphi^n(x+\psi_nh)$, " ψ_n " " 0 and ψ_{n-1}. (g)

. . . .

$\dfrac{\varphi^{2n-1}(x+\psi_{2n-1}h)-\varphi^{2n-1}x}{\psi_{2n-1}h}=\varphi^{2n}(x+\psi_{2n}h)$, " ψ_{2n}" " 0 and ψ_{2n-1}. (l)

In general, $\psi_1, \psi_2, \ldots$ are functions of both x and h, and do not reduce to zero when $h=0$. For, each being alwys positive, and never exceeding unity, neither can contain h as a factor in every term and consequently cannot become zero when $h=0$. Let $\psi'_1, \psi'_2, \psi'_3, \ldots$ be the values to which $\psi_1, \psi_2, \psi_3, \ldots$ respectively reduce, when $h=0$.

In (b), replace ψ_1 in the denominator of the left member by ψ'_1, and denote what the right member becomes by $\varphi''(x+\psi''_2h)$. Then $[\varphi'(x+\psi_1h)—\varphi'x]\div\psi'_1h=\varphi''(x+\psi''_2h)$. Let ψ''_2 be substituted for ψ_2 in the left member of (c), and denote what the right member becomes by $\varphi'''(x+\psi'''_3h)$. Then $[\varphi''(x+\psi''_2h)—\varphi''x]\div\psi''_2h=\varphi'''(x+\psi'''_3h)$. In this, let ψ'_2 be substituted for ψ''_2 in the denominator of the left member and denote what the right member becomes by $\varphi'''(x+\psi''_3h)$. Then $[\varphi''(x+\psi''_2h)—\varphi''x]\div\psi'_2h=\varphi'''(x+\psi''_3h)$. Continuing this process, we shall finally get $[\varphi^{n-1}(x+\psi''_{n-1}h)—\varphi^{n-1}x]\div\psi'_{n-1}h=\varphi^n(x+\psi''_nh)$. Collecting the several results, we have with (a), $[\varphi(x+h)—\varphi x]\div h=\varphi'(x+\psi_1h)$, $[\varphi'(x+\psi_1h)—\varphi'x]\div\psi'_1h=\varphi''(x+\psi''_2h)$, $\varphi''(x+\psi''_2h)—\varphi''x]\div\psi'_2h=\varphi'''(x+\psi''_3h), \ldots$ $[\varphi^{n-1}(x+\psi''_{n-1}h)—\varphi^{n-1}x]\div\psi'_{n-1}h=\varphi^n(x+\psi''_nh)$.

Eliminating $\varphi'(x+\psi_1h)$, $\varphi''(x+\psi''_2h), \ldots \varphi^{n-1}(x+\psi''_{n-1}h)$, we get

$$\varphi(x+h)=\varphi x+h\varphi'x+\psi'_1h^2\varphi''x+\psi'_1\psi'_2h^3\varphi'''x+\ldots\psi'_1\psi'_2\ldots\psi'_{n-1}h^n \times\varphi^n(x+\psi''_nh). \quad (3)$$

In (3), $\psi'_1, \psi'_2, \ldots \psi'_{n-1}$ are independent of h, and ψ''_n is some quantity which is in general a function of both x and h. By differentiating (3) n times with respect to h, we obtain the following equations:

$$\varphi'(x+h)=\varphi'x+2\psi'_1h\varphi''x+3\psi'_1\psi'_2h^2\varphi'''x+4\psi'_1\psi'_2\psi'_3h^3\varphi''''x+\ldots n\psi'_1\psi'_2 \ldots\psi'_{n-1}h^{n-1}\varphi^n(x+\psi''_nh)+\psi'_1\psi'_2\ldots\psi'_{n-1}h^n[d\varphi^n(x+\psi''_nh)]\div dh.$$

$$\varphi''(x+h)=2\psi'_1\varphi''x+2.3\psi'_1\psi'_2h\varphi'''x+3.4\psi'_1\psi'_2\psi'_3h^2\varphi''''x+\ldots(n—1)n\psi'_1\psi'_2 \ldots\psi'_{n-1}h^{n-2}\varphi^n(x+\psi''_nh)+2n\psi'_1\psi'_2\ldots\psi'_{n-1}h^{n-1}[d\varphi^n(x+\psi'_n{}'h)\div dh]+\ldots$$

$$\varphi'''(x+h)=2.3\psi'_1\psi'_2\varphi'''x+2.3.4\psi'_1\psi'_2\psi'_3h\varphi''''x+\ldots.(n-2)(n-1)n\psi'_1\psi'_2\ldots \psi'_{n-1}h^{n-3}\varphi^n(x+\psi''_nh)+3(n-1)n\psi'_1\psi'_2\ldots\psi'_{n-1}h^{n-2}[d\varphi^n(x+\psi''_nh)\div dh]+\ldots$$

. . . .

$$\varphi^n(x+h)=1.2.3\ldots n\psi'_1\psi'_2\ldots\psi'_{n-1}\varphi^n(x+\psi''_nh)+n.2.3\ldots n\psi'_1\psi'_2\ldots \psi'_{n-1}h[d\varphi^n(x+\psi''_nh)\div dh]+\ldots \quad (4)$$

In these equations, let $h=0$. Then we obtain $\varphi'x=\varphi'x$; $\varphi''x=2\psi'_1\varphi''x$. $\therefore\ \psi'_1=\frac{1}{2}$. $\varphi'''x=2.3\psi'_1\psi'_2\varphi'''x$, $\therefore\ \psi'_2=\frac{1}{3}$. $\varphi''''x=2.3.4\psi'_1\psi'_2\psi'_3\varphi''''x$,

$\therefore\ \psi_3 = \frac{1}{4}$; and in general, $\psi'_{n-1} = 1 \div n$. These values of $\psi'_1 \ldots \psi'_{n-1}$ in (3) and (4) give respectively;

$$\varphi\,(x+h) = \varphi x + \mathrm{h}\varphi' x + \frac{h^2}{2}\varphi'' x + \frac{h^3}{2.3}\varphi''' x \ldots \frac{h^n}{2\,.\,3\ldots n}\varphi^n(x+\psi''_n h). \quad (5)$$

$$\varphi^n(x+h) = \varphi^n(x+\psi''_n h) + nh\frac{d\varphi^n(x+\psi''_n h)}{dh} + \ldots \quad (6)$$

From (6) we find that $\varphi^n(x+\psi''_n h) < \varphi^n(x+h)$. $\therefore\ \psi''_n < 1$. If m be any integer between n and $2n$, then (5) can become

$$\varphi(x+h) = \varphi x + h\varphi' x + \frac{h^2}{2}\varphi'' x + \ldots \frac{h^n}{\underline{|n}}\varphi^n x + \frac{h^{n+1}}{\underline{|n+1}}\varphi^n x + \ldots \frac{h^m}{\underline{|m}} \times \varphi^m(x+\psi''_m h). \quad (7)$$

$$\therefore\ \varphi^n(x+\psi''_n h) = \varphi^n x + \frac{h}{n+1}\varphi^{n+1}x + \frac{h^2}{(n+1)(n+2)}\varphi^{n+2}x + \ldots \frac{h^m}{(n+1)(n+2)..m} \times \varphi^m(x+\psi''_m h).$$

$\therefore\ \varphi^n(x+\psi''_2 h) > \varphi^n x$, and ψ''_n is positive; but $\psi''_n < 1$, therefore it is a positive proper fraction. From (5) and (7) we also find that $\psi''_n > \psi''_m$. Hence we have $1 > \psi_1 > \psi''_2 > \psi''_3 > \psi''_4 \ldots \psi''_{u-1} > \psi''_n$. Hence also, we can make ψ''_n as small as we please, *provided* we can make n large enough and still have $\varphi^n x$ finite and continuous.

With this condition, we can make $\varphi^n(x+\psi''_n h)$ differ from $\varphi^n x$ by as small an amount as may be desired, and we can write:

$$\varphi(x+h) = \varphi x + h\varphi' x + \frac{h^2}{2}\varphi'' x + \ldots\ldots + \frac{h^n}{\underline{|n}}\varphi^n x, \quad (8)$$

which gives the value of $\varphi(x+h)$ to any desired degree of accuracy when n is large enough.

Taylor's formula can be derived directly from equations (a), (b), (c), . . . as follows:—Because $1 > \psi_1 > \psi_2 \ldots\ldots > \psi_{2n-1} > \psi_{2n}$, we can make $\varphi^{2n}(x+\psi_{2n}h)$ differ from $\varphi^{2n}x$ by as small an amount as we please, *provided* we can make n large enough and still have $\varphi^{2n}x$ finite and continuous. Hence eliminating $\varphi'(x+\psi_1 h)$, $\varphi''(x+\psi_2 h)$ . . . . we get

$$\varphi(x+h) = \varphi x + h\varphi' x + \psi_1 h^2\varphi'' x + \psi_1\psi_2 h^3\varphi''' x + \ldots \psi_1\psi_2 \ldots \psi_{n-1}h^n\varphi^n x + \psi_1\psi_2 \ldots \psi_n h^{n+1}\varphi^{n+1}x + \ldots \psi_1\psi_2 \ldots \psi_{2n-1}h^{2n}\varphi^{2n}x, \quad (9)$$

where the values of $\varphi(x+h)$, will be expressed to any desired degree of accuracy, when n is large enough.

Forming the differential coefficients of (9), regarding h as constant; and again regarding x as constant, and subtracting the second result from the first, we obtain,

$$0 = (1-2\psi_1)h\varphi'' x + \left(\frac{d\psi_1}{dx} - \frac{d\psi_1}{dh}\right)h^2\varphi'' x + (\psi_1 - 3\psi_1\psi_2)h^2\varphi''' x + \left(\frac{d\psi_1}{dx} - \frac{d\psi_1}{dh}\right) \times \psi_2 h^3\varphi''' x + \psi_1\left(\frac{d\psi_2}{dx} - \frac{d\psi_2}{dh}\right)h^3\varphi''' x + (\psi_1\psi_2 - 4\psi_1\psi_2\psi_3)h^3\varphi'''' x + \ldots \quad (10)$$

If in (10) we put $1-2\psi_1=0$, $\psi_1-3\psi_1\psi_2=0$, $\psi_1\psi_2-4\psi_1\psi_2\psi_3=0, \ldots\ldots$ then, $\psi_1=\frac{1}{2}$, $\psi_2=\frac{1}{3}$, $\psi_3=\frac{1}{4}, \ldots \psi_{2n-1}=1\div 2n$; and these values being constants,

$$\frac{d\psi_1}{dx},\ \frac{d\psi_1}{dh},\ \frac{d\psi_2}{dx},\ \frac{d\psi_2}{dh},\ldots$$

each becomes zero, and equation (10) is *completely satisfied.* Hence, $\psi_1=\frac{1}{2}$, $\psi_2=\frac{1}{3}$, $\psi_3=\frac{1}{4}, \ldots\ldots \psi_{2n-1}=1\div 2n$, are always included among the possible values of these fractions in (9), and give

$$\varphi(x+h)=\varphi x+h\varphi' x+\frac{h^2}{2}\varphi'' x+\ldots\frac{h^n}{\underline{|n}}\varphi^n x+\frac{h^{n+1}}{\underline{|n+1}}\varphi^{n+1}x+\ldots\frac{h^{2n}}{\underline{|2n}}\varphi^{2n}x+\ldots \quad \ldots (11)$$

We can also have from (*a*), (*b*), (*c*), . . .

$$\varphi(x+h)=\varphi x+h\varphi' x+\psi_1 h^2\varphi'' x+\psi_1\psi_2 h^3\varphi''' x+\ldots\psi_1\psi_2\ldots\psi_{n-1}h^n \times\varphi^n(x+\psi_n h),$$

in which, giving $\psi_1, \psi_2 \ldots$ their foregoing values and denoting by ψ''_n what ψ_n may in consequence become, we get equation (5) in which ψ''_n has been shown to be a positive proper fraction.

NOTE ON THE HISTORY OF THE METHOD OF LEAST SQUARES.

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I.

IN the March number of this Journal I stated the first proof of the Method of Least Squares to have been given by Dr. Robert Adrain in 1808 and the second by Gauss in 1809. At that time I had not seen Adrain's original paper and my information concerning it had been entirely derived from an account and partial reprint given by Prof. Cleveland Abbe in the *Amer. Jour. Sci.*, 1871, Vol. I, pp. 411–415. Having lately been able to consult a copy of *The Analyst or Mathematical Museum* in which Adrain's paper was published I found that on pages 96 and 97, to which Prof. Abbe makes no reference, there is given a second deduction of the law of facility of error of an entirely different nature from that presented on pages 93–95. As this is a matter of considerable historical interest and as *The Analyst* for 1808 is quite rare I give the proof in Adrain's own words,